7. Appendix

7.1. Computation of the equilibrium point

In order to calculate the equilibrium point of the system (1), all of the equations must first be set equal to zero, and then solved.

$$\alpha_M E_M - \mu_M A_M = 0, \tag{9a}$$

$$\alpha_F E_F - \mu_F A_F = 0, \tag{9b}$$

$$-\alpha_M E_M + pr\left(1 - \frac{A_M + A_F}{K}\right)\left(\frac{bA_F A_M}{A_F + aA_M}\right) - \mu_E E_M = 0, \tag{9c}$$

$$-\alpha_F E_F + (1-p)r\left(1 - \frac{A_M + A_F}{K}\right)\left(\frac{bA_F A_M}{A_F + aA_M}\right) - \mu_E E_F = 0. \tag{9d}$$

7.1.1. Non-zero equilibrium point

First we add the four equations together to get

$$rbA_M\left(1 - \frac{A_M + A_F}{K}\right)\left(\frac{A_F}{A_F + aA_M}\right) = \mu_E E_M + \mu_M A_M + \mu_E E_F + \mu_F A_F. \tag{10}$$

From (9a) and (9b), we know

$$E_M = \frac{\mu_M}{\alpha_M} A_M, \qquad E_F = \frac{\mu_F}{\alpha_F} A_F, \tag{11}$$

Substituting (11) into (10), we obtain

$$rbA_M\left(1 - \frac{A_M + A_F}{K}\right)\left(\frac{A_F}{A_F + aA_M}\right) = \frac{\alpha_M\mu_M + \mu_M\mu_E}{\alpha_M} A_M + \frac{\alpha_F\mu_F + \mu_F\mu_E}{\alpha_F} A_F \tag{12}$$

Equation (9c) can be rewritten as

$$rbA_M\left(1 - \frac{A_M + A_F}{K}\right)\left(\frac{A_F}{A_F + aA_M}\right) = \frac{\alpha_M + \mu_E}{p} E_M \tag{13}$$

and substituting (11) into (13) we get

$$rbA_M\left(1 - \frac{A_M + A_F}{K}\right)\left(\frac{A_F}{A_F + aA_M}\right) = \frac{(\alpha_M + \mu_E)\mu_M}{\alpha_M p} A_M \tag{14}$$

Using equations (10) and (14) we obtain

$$\frac{\alpha_M\mu_M + \mu_M\mu_E}{\alpha_M} A_M + \frac{\alpha_F\mu_F + \mu_F\mu_E}{\alpha_F} A_F = \frac{(\alpha_M + \mu_E)\mu_M}{\alpha_M p} A_M \tag{15}$$

Rewriting (15), we obtain the relationship between A_M and A_F .

$$HA_M = GA_F$$

That is $\frac{A_F}{A_M} = \frac{H}{G}$ and $A_M + A_F = \left(1 + \frac{H}{G}\right) A_M$, where $H = \frac{(\mu_M \mu_E + \mu_M \alpha_M)(p-1)}{p \alpha_M}$ and $G = \frac{\mu_F \mu_E + \mu_F \alpha_M}{\alpha_F}$, substituting into (15), we obtain

$$\frac{r}{K}\left(K - \left(1 + \frac{H}{G}\right) A_M\right)\left(\frac{bH}{H + aG}\right) A_M = \frac{(\alpha_M + \mu_E)\mu_M}{\alpha_M p} A_M$$

Solving non-zero solution of above equation give us,

$$A_M^* = \frac{(K - Q)G}{H + G}$$

Where $Q = \frac{(\mu_E + \alpha_M)\mu_M (H + a)K}{\alpha_M p r b H}$. From de relationship between A_M , A_F , E_M and E_F the non-zero equilibrium point is

$$I^* \left(A_M^*, \frac{H}{G} A_M^*, \frac{\mu_M}{\alpha_M} A_M^*, \frac{H \mu_F}{G \alpha_F} A_M^*\right).$$

7.2. Existence condition

The existence condition for I^* existence condition is

$$A_M^* = \frac{(K-Q)G}{H+G} > 0 \tag{16}$$

$H, G, Q > 0$. Then $A_M > 0$ if and only if $K > Q$, which is equivalent to

$$p > \frac{(\alpha_M + \mu_E)\mu_M (H + aG)}{r b H \alpha_M} \tag{17}$$

By inserting this simplification into our expression, we can rewrite (17) and define new expressions.

Refine, $J = \frac{(\mu_E + \alpha_M)\mu_M}{\alpha_M b r}$ and $D = \frac{(\mu_E + \alpha_F) a \mu_F}{\alpha_F b r}$, then (17)is rewritten as

$$p > J + D \frac{p}{1-p} \tag{18}$$

The inequality is solved in terms of p producing the next quadratic expression

$$p^2 + (D - J - 1)p + J < 0, \tag{19}$$

where p_1 and p_2 are the solutions of $p^2 + (D - J - 1)p + J = 0$, namely

$$p_1 = \frac{J - D + 1 - \sqrt{\Delta}}{2}, \qquad p_2 = \frac{J - D + 1 + \sqrt{\Delta}}{2}.$$

Finally, the existence condition is

$$\Delta = (D - J - 1) - 4J > 0 \text{ and } p_1 < p < p_2. \tag{20}$$

7.3. Range for temperature

By substituting function (7) into inequality (4), we get

$$p_1 < p(T) < p_2$$

$$p_1 < \frac{1}{1 + e^{-\frac{1}{S}(\beta - T)}} < p_2$$

$$\frac{1}{p_1} < 1 + e^{-\frac{1}{S}(\beta - T)} < \frac{1}{p_2}$$

$$\frac{1}{p_1} - 1 < e^{-\frac{1}{S}(\beta - T)} < \frac{1}{p_2} - 1$$

$$ln\left(\frac{1}{p_1} - 1\right) < -\frac{1}{S}(\beta - T) < ln\left(\frac{1}{p_2} - 1\right)$$

$$ln\left(\frac{1}{p_1} - 1\right)S < -(\beta - T) < ln\left(\frac{1}{p_2} - 1\right)S$$

$$ln\left(\frac{1}{p_2} - 1\right)S + \beta < T < ln\left(\frac{1}{p_1} - 1\right)S + \beta$$